Ahlfors-Weill extensions in several complex variables

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Abstract. We derive an Ahlfors-Weill type extension for a class of holomorphic mappings defined in the ball $\mathbb{B}^n$, generalizing the formula for Nehari mappings in the disk. The class of mappings holomorphic in the ball is defined in terms of the Schwarzian operator. Convexity relative to the Bergman metric plays an essential role, as well as the concept of a weakly linearly convex domain. The extension outside the ball takes values in the projective dual to $\mathbb{C}^n$, that is, in the set of complex hyperplanes.

1. Introduction

The purpose of this paper is to obtain an explicit extension to $\mathbb{C}^n$ for a class of biholomorphic mappings F defined in the unit ball $\mathbb{B}^n$, which parallels the Ahlfors-Weill construction derived for certain univalent mappings of the disk [2]. The analysis adapts to several variables the notion of conformal barycenter used in [6], and leads to an extension E_F of F that assumes values in the projective dual to $\mathbb{C}^n$. More precisely, for $|z| > 1$, $E_F(z)$ is a certain complex hyperplane disjoint from the closure $F(\mathbb{B}^n)$, and the extension is a "homeomorphism" in the sense that $E_F(z)$ depends in a continuous and injective way on z. Moreover, $E_F(z)$ approaches a support hyperplane of $\overline{F(\mathbb{B}^n)}$ as $|z| \to 1^+$, while $E_F(z)$ leaves any compact subset as $|z| \to \infty$. As a consequence of the analysis, $F(\mathbb{B}^n)$ is shown to be weakly linearly convex, that is, a domain disjoint from a complex hyperplane containing any given point on the boundary (see, *e.g.*, [15], [3]). One can consequently visualize the extension as gluing $F(\mathbb{B}^n)$ to a complementary domain in the projective dual through the matching of a boundary point with a supporting complex hyperplane.

The classes of mappings F considered in this paper are defined in terms of the Schwarzian derivative, and in our setting we appeal to the work of T. Oda for a generalization of the classical operator in one variable. The full extent of our results require the assumption of quasiregularity in addition to a Schwarzian bound. The hypothesis of quasiregularity appears also in other related extensions in several variables, such as the use of Loewner chains for the generalization of the Becker

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univalence crietrion or for extending other classes of holomorphic mappings (see [27] and, *e.g.*, [11], [12]).

The higher dimensional Schwarzian splits into differential operators $\mathcal{S}F$ and $\mathcal{S}_0F$ of order two and three, respectively, a feature that is also present in other formulations of this operator in several variables (see, *e.g.*, [18], [19], [21]). The second author has developed important properties of Oda's Schwarzian, in particular, in connection with invariance and injectivity criteria [13], [14]. Our work relies in a significant way on the behavior of a real valued function (*density function*) associated in a canonical way with a locally biholomorphic mapping. The density function becomes convex relative to the Bergman metric once adequate bounds on the Schwarzian are in force. For holomorphic mappings of one complex variable, the real valued function is simply the square root of the Poincaré density of the image domain. The treatment in several variables must deal with technical difficulties absent in one variable, and which arise from first order derivatives in the linear system associated with the Schwarzian. These terms can be controlled by the (trace) order of the family $\mathcal{F}_\alpha$ (see Definition 2.4 ahead), and indirectly, by the norm of the Schwarzian.

Throughout the paper we consider Möbius shifts of a mapping $F = (f_1, \ldots, f_n)$ of the form

$$T \circ F = (l_1(F)/l_0(F), \ldots, l_n(F)/l_0(F)) \ , \ l_i(F) = a_{i0} + a_{i1}f_1 + \cdots + a_{in}f_n \, .$$

The appearance of the hypersurfaces of singularities $l_0(F) = 0$ within the ball destroys the order of the family, and we are forced to modify the density function and analyze its behavior regarding convexity away from the hypersurfaces.

As a byproduct of the analysis on convexity, we are able to establish an injectivity criterion solely in terms of $\mathcal{S}F$. This is more satisfactory than the condition derived in [14] that requires an additional bound on $\mathcal{S}_0F$. Convexity also renders distortion theorems for appropriately normalized mappings in the classes treated, with estimates on the jacobian that are better than those obtained from the order of the family [28].

In [7], we adapt the variational method introduced in [29] to estimate the order of the family $\mathcal{F}_\alpha$ in terms of α and the dimension n. As a consequence, we are able to provide, in terms of n, a range for the Schwarzian norm that ensures convexity, and later on, univalence and extensions.

In Section 2 of the paper we present the definition and basic properties of the Schwarzian derivative in several variables, including the important Lemma 2.1 never stated in the work of Oda. In Section 3, we introduce both the density and the modified density function, together with the results regarding convexity. The analysis renders Theorem 3.9, which states that for $\alpha_0 \leq cn^{-\frac{3}{2}}$, c an absolute constant, then functions in $\mathcal{F}_{\alpha_0}$ are univalent in $\mathbb{B}^n$.

In Section 4 we collect several results on the jacobian, distortion and continuity up to the boundary that are obtained from convexity and quasiregularity. Quasiregular mappings in the class $\mathcal{F}_{\alpha_0}$ extend continuously to $\overline{\mathbb{B}^n}$ and remain univalent there. We also lay out the important process of normalization on which the method of conformal barycenter is based. In particular, we show that for $F \in \mathcal{F}_{\alpha_0}$ and every $z_0 \in \mathbb{B}^n$ the image $F(\mathbb{B}^n)$ omits a certain hyperplane $\mathcal{H}(z_0)$. This hyperplane represents the *conformal barycenter* of F relative to z_0, and reduces to a single point when $n = 1$. The explicit formula for $\mathcal{H}(z_0)$ is consistent with the expression for the Ahlfors-Weill extension when $n = 1$. When $F \in \mathcal{F}_{\alpha_0}$ is quasiregular, then $\overline{F(\mathbb{B}^n)} \cap \mathcal{H}(z_0) = \emptyset$, while $\mathcal{H}(z_0)$ approaches a hyperplane of support of $F(\mathbb{B}^n)$ as $|z_0| \to 1$. From this, the images are shown to be weakly linearly convex. With the notation $z^* = z/|z|^2$ we obatin in Theorem 5.1 the extension

$$E_F(z) = \begin{cases} F(z) & , \quad |z| \leq 1 \\ \\ \mathcal{H}(z^*) & , \quad |z| > 1 \end{cases} \tag{1.1}$$

which glues $F(\overline{\mathbb{B}^n})$ with a complementary domain in the projective dual of $\mathbb{C}^n$.

2. Preliminaries

Let $F : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a locally biholomorphic mapping defined on a domain Ω. In [25] T.Oda defined a family of Schwarzian derivatives of $F = (f_1, \ldots, f_n)$ as

$$S^k_{ij}F = \sum_{l=1}^{n} \frac{\partial^2 f_l}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial f_l} - \frac{1}{n+1} \left(\delta^k_i \frac{\partial}{\partial z_j} + \delta^k_j \frac{\partial}{\partial z_i} \right) \log JF \,, \tag{2.1}$$

where $i, j, k = 1, 2, \ldots, n$, $JF = \det(DF)$ is the jacobian determinant of the diferential DF and δ^k_i are the Kronecker symbols. For $n > 1$ the Schwarzian derivatives have the following properties:

$$S^k_{ij}F = 0 \quad \text{for all} \quad i, j, k = 1, 2, \ldots, n \quad \text{iff} \quad F(z) = M(z) \,, \tag{2.2}$$

for some Möbius transformation

$$M(z) = \left(\frac{l_1(z)}{l_0(z)}, \ldots, \frac{l_n(z)}{l_0(z)} \right) ,$$

where $l_i(z) = a_{i0} + a_{i1}z_1 + \cdots + a_{in}z_n$ with $\det(a_{ij}) \neq 0$. For a composition

$$S^k_{ij}(G \circ F)(z) = S^k_{ij}F(z) + \sum_{l,m,r=1}^{n} S^r_{lm}G(w) \frac{\partial w_l}{\partial z_i} \frac{\partial w_m}{\partial z_j} \frac{\partial z_k}{\partial w_r} \,, \; w = F(z) \,. \tag{2.3}$$

Thus, if G is a Möbius transformation then $S^k_{ij}(G\circ F)=S^k_{ij}F$. The $S^0_{ij}F$ coefficients are given by

$$S^0_{ij}F(z)=(JF)^{1/(n+1)}\left(\frac{\partial^2}{\partial z_i\partial z_j}(JF)^{-1/(n+1)}-\sum_{k=1}^n\frac{\partial}{\partial z_k}(JF)^{-1/(n+1)}S^k_{ij}F(z)\right).$$

One can recover a mapping from its Schwarzian components. Consider the following overdetermined system of partial differential equations,

$$\frac{\partial^2 u}{\partial z_i\partial z_j}=\sum_{k=1}^n P^k_{ij}(z)\frac{\partial u}{\partial z_k}+P^0_{ij}(z)u\ ,\quad i,j=1,2,\ldots,n\,, \tag{2.4}$$

where $z=(z_1,z_2,...,z_n)\in\Omega$ and $P^k_{ij}(z)$ are holomorphic functions in Ω, for $k=0,\ldots,n$ and $i,j=1,\ldots,n$. The system (2.4) is called *completely integrable* if there are $n+1$ (maximum) linearly independent solutions, and is said to be in *canonical form* (see [31]) if the coefficients satisfy

$$\sum_{j=1}^n P^j_{ij}(z)=0\ ,\quad i=1,2,\ldots,n\,.$$

Oda proved that (2.4) is a completely integrable system in canonical form if and only if $P^k_{ij}=S^k_{ij}F$ for a locally boholomorphic mapping $F=(f_1,\ldots,f_n)$, where $f_i=u_i/u_0$ for $1\le i\le n$ and $u_0,u_1,\ldots,u_n$ is a set of linearly independent solutions of the system. A simple corollary of this is that F fails to be univalent in Ω iff there exists a set $u_1,\ldots,u_n$ of linearly independent solutions of (2.4) with $P^k_{ij}=S^k_{ij}F$ which vanish at two distinct points in Ω [14]. For a given mapping F, $u_0=(JF)^{-\frac{1}{n+1}}$ is always a solution of (2.4) with $P^k_{ij}=S^k_{ij}F$. The following result not stated in the work of Oda will be important in the rest of the paper.

Lemma 2.1. *Let u be a solution of a completely integrable system of the form (2.4) with $P^k_{ij}=S^k_{ij}F$ for some locally biholomorphic mapping F defined in Ω. Then there exists a Möbius transformation T such that $u=(JG)^{-\frac{1}{n+1}}$ for $G=T\circ F$.*

Proof. We write $F=(u_1/u_0,\ldots,u_n/u_0)$ for $n+1$ linearly independent solutions $u_0,u_1,\ldots,u_n$ of (2.4) with $u_0=(JF)^{-\frac{1}{n+1}}$. Then $u=b_0u_0+b_1u_1+\cdots+b_nu_n$ for some unique constants b_i. A simple calculation shows that $(JT)^{-\frac{1}{n+1}}=a_0+a_1w_1+\cdots+a_nw_n=l_0(w)$ whenever T is a Möbius transformation of the form $(w_1/l_0(w),\ldots,w_n/l_0(w))$. Then

$$\begin{aligned}(J(T\circ F))^{-\frac{1}{n+1}} &= (JT(F))^{-\frac{1}{n+1}}(JF)^{-\frac{1}{n+1}}\\ &= (a_0+a_1f_1+\cdots+a_nf_n)u_0\\ &= a_0u_0+a_1u_1+\cdots+a_nu_n\,,\end{aligned}$$

hence it suffices to choose T with the property that $(JT)^{-\frac{1}{n+1}} = b_0+b_1z_1+\cdots+b_nz_n$. Note that the zero set of u is given by the hypersurface $a_0 + a_1f_1 + \cdots + a_nf_n = 0$, that is, exactly the set where G becomes singular. □

Remark: We will say that the Möbius transformation T as in the Lemma 2.1 is an *inversion* with respect to the hyperplane $l_0(w) = 0$.

We recall the following definitions from [13], where the individual Schwarzians $S^k_{ij}F$ are grouped together as an operator.

Definition 2.2. For each $k = 1, \ldots, n$ we let $\mathbb{S}^kF$ be the $n \times n$ matrix

$$\mathbb{S}^kF = (S^k_{ij}F)\,, \quad i, j = 1, \ldots, n\,.$$

Definition 2.3. We define the *Schwarzian derivative operator* as the bilinear mapping $\mathcal{S}F(z) : T_z\Omega \to T_z\Omega$ given by

$$\mathcal{S}F(z)(\vec{v}) = \left(\vec{v}^{\,t}\mathbb{S}^1F(z)\vec{v}\,,\, \vec{v}^{\,t}\mathbb{S}^2F(z)\vec{v}\,,\, \ldots, \vec{v}^{\,t}\mathbb{S}^nF(z)\vec{v}\right)\,,$$

where $\vec{v} \in T_z\Omega$.

The operator $\mathcal{S}F(z)$ inherits a norm from the metric in $T_z\Omega$:

$$\|\mathcal{S}F(z)\| = \sup_{\|\vec{v}\|=1} \|\mathcal{S}F(z)(\vec{v}\,)\|\,, \tag{2.5}$$

and finally, we let

$$||\mathcal{S}F|| = \sup_{z\in\Omega} ||\mathcal{S}F(z)||\,. \tag{2.6}$$

Our interest is to study certain classes of locally biholomorphic mappings F defined in the unit ball $\mathbb{B}^n$. The Bergman metric g_B on $\mathbb{B}^n$ is the hermitian product defined by

$$g_{ij}(z) = \frac{n+1}{(1-|z|^2)^2}\,\left[(1-|z|^2)\delta_{ij} + \bar{z}_iz_j\right]. \tag{2.7}$$

Is well known that the automorphism group of the ball are the transformations given by

$$\sigma(z) = \frac{Az+B}{Cz+D}\,,$$

where A is $n \times n$, B is $n \times 1$, C is $1 \times n$ and D is 1×1 with

$$\begin{aligned} A^t\overline{A} - C^t\overline{C} &= \mathrm{Id}\,, \\ |D|^2 - B^t\overline{B} &= 1\,, \\ A^t\overline{B} - C^t\overline{D} &= 0\,, \end{aligned}$$

and that such automorphisms are isometries in the Bergman metric (see, *e.g.*, [17]).

As a consequence of this and of the chain rule (2.3), it was shown in [13] that

$$\|\mathcal{S}(F\circ\sigma)(z)\| = \|\mathcal{S}F(\sigma(z))\|\,,$$

and hence

$$\|\mathcal{S}F\| = \|\mathcal{S}(F\circ\sigma)\|\,. \tag{2.8}$$

More generally, the Schwarzian norm is preserved under Möbius maps between domains, that is, if σ is Möbius and $\Omega_2 = \sigma(\Omega_1)$ then $\|\mathcal{S}F\|_2 = \|\mathcal{S}(F\circ\sigma)\|_1$ whenever F is locally biholomorphic in Ω_2.

Definition 2.4. We define the class $\mathcal{F}_\alpha$ as

$$\mathcal{F}_\alpha = \{\, F:\mathbb{B}^n\to\mathbb{C}^n \mid F \text{ locally biholomorphic}\,,\ F(0)=0\,,\ DF(0)=\mathrm{Id}\,,\ \|\mathcal{S}F\|\le\alpha\,\}\,.$$

In light of (2.8), $\mathcal{F}_\alpha$ is a linearly invariant family, which is also compact [13]. The order of a linearly invariant family is defined by

$$\sup_{F\in\mathcal{F}_\alpha}\ \sup_{|w|=1}\ \frac{1}{2}\left|\sum_{i,j=1}^{n}\frac{\partial^2 f_j}{\partial z_i\partial z_j}(0)w_i\right|\,, \tag{2.9}$$

which is finite for compact linearly invariant families (see, *e.g.*, [28]).

3. Convexity and Univalence

Let $F\in\mathcal{F}_\alpha$, and consider a solution u of the integrable system

$$\frac{\partial^2 u}{\partial z_i\partial z_j} = \sum_{k=1}^{n} S_{ij}^k F\,\frac{\partial u}{\partial z_k} + S_{ij}^0 F u\,,\quad i,j=1,2,\ldots,n\,. \tag{3.1}$$

We define

$$\lambda_u(z) = \frac{|u(z)|}{\sqrt{1-|z|^2}}\,.$$

The function λ_u is well-defined and real analytic away from the zero set of u. According to Lemma 2.1, a nowhere vanishing solution of the system is of the form $u = (JF)^{-\frac{1}{n+1}}$ for some $F\in\mathcal{F}_\alpha$; the corresponding function λ_u will be denoted by λ_F.

Lemma 3.1. *Let σ be an automorphism of $\mathbb{B}^n$ and $F\in\mathcal{F}_\alpha$. Then*

$$\lambda_{F\circ\sigma} = \lambda_F\circ\sigma\,.$$

Proof. We have that

$$\lambda_{F\circ\sigma}(z) = |J(F\circ\sigma)(z)|^{-1/n+1}\,(1-|z|^2)^{-1/2}\,,$$

but $|J\sigma(z)| = \left(\dfrac{1-|\sigma(z)|^2}{1-|z|^2}\right)^{\frac{n+1}{2}}$, therefore

$$\lambda_{F\circ\sigma}(z) = |JF(\sigma(z))|^{-1/n+1}\left(\frac{1-|\sigma(z)|^2}{1-|z|^2}\right)^{-1/2}(1-|z|^2)^{-1/2} = \lambda_F(\sigma(z))\,.$$

□

Since $\mathcal{F}_\alpha$ is a compact family, it has finite order, or equivalently,

$$\mathcal{A}_\alpha = \sup_{F\in\mathcal{F}_\alpha}|\nabla(JF)(0)| < \infty\,.$$

For the equivalence of the above with the definition given in (2.9), see [13].

Lemma 3.2. *Let $F \in \mathcal{F}_\alpha$ and suppose u is a solution of the completely integrable system (2.4) with $P_{ij}^k = S_{ij}^k F$. If $u \neq 0$ in $\mathbb{B}^n$ then the function $\varphi(x) = |u(x,0,\ldots,0)|$ satisfies*

$$\varphi''(x) + \frac{\delta}{(1-x^2)^2}\varphi(x) \geq 0\,,$$

with

$$\delta = \sqrt{n+1}(\mathcal{A}_\alpha+1)\alpha + C(n,\alpha)\,,$$

where $\mathcal{A}_\alpha$ is the order of $\mathcal{F}_\alpha$ and

$$C(n,\alpha) = \left(4n^2+2n-2+\frac{n+1}{n-1}\right)\alpha^2 + \left(4\sqrt{n+1}+8\frac{\sqrt{n+1}}{n-1}\right)\alpha\,.$$

Proof. By Lemma 2.1, there exists a function $F \in \mathcal{F}_\alpha$ such that $u = (JF)^{-1/n+1}$. Since $\mathcal{F}_\alpha$ ia a compact family it has finite order

$$\mathcal{A}_\alpha = \sup_{F\in\mathcal{F}_\alpha}|\nabla\log(JF)(0)| < \infty\,.$$

(For the equivalence of the above with the definition given in (2.9), see [13].) Let σ be an automorphism of $\mathbb{B}^n$, $G = [DF(\zeta)]^{-1}[D\sigma(0)]^{-1}(F\circ\sigma(z) - F(\zeta))$ where $\sigma(0) = \zeta$. Thus $G \in \mathcal{F}_\alpha$ and $v = (JG)^{-1/n+1} = u\cdot(J\sigma)^{-1/n+1} = u\cdot w$, and

$$JF(\zeta)J\sigma(0)\nabla v(z) = \nabla u(\sigma(z))D\sigma(z)w(z) + u(\sigma(z))\nabla w(z)\,.$$

At $z = 0$, we have that

$$\nabla u(\zeta)D\sigma(0) = \left(\frac{\nabla v}{v}(0) - \frac{\nabla w}{w}(0)\right)\,,$$

where the euclidean norm satisfies

$$\left(\left|\frac{\partial u}{\partial z_1}(\zeta)(1-|\zeta|^2)\right|^2 + \sum_{k=2}^n\left|\frac{\partial u}{\partial z_k}(\zeta)\sqrt{1-|\zeta|^2}\right|^2\right)^{1/2} \leq (\mathcal{A}_\alpha+1)|u(\zeta)|\,. \tag{3.2}$$

If $X=(x,0,\dots,0)$ and $\vec{a}=((1-x^2)/\sqrt{n+1},0,\dots,0)$ then

$$\|\mathcal{S}F(X)(\vec{a}\,)\|^2=(n+1)\left(\frac{|S^1_{11}F(X)|^2(1-x^2)^4}{(n+1)^2(1-x^2)^2}+\sum_{k=2}^{n}\frac{|S^k_{11}F(X)|^2((1-x^2)^4}{(n+1)^2(1-x^2)}\right)\leq\alpha^2\,,$$

and using equation (3.2) we obtain that

$$\left|S^1_{11}F(X)\frac{\partial u}{\partial z_1}(X)+\sum_{k=2}^{n}S^k_{11}F(X)\frac{\partial u}{\partial z_k}(X)\right|\leq\frac{\sqrt{n+1}(\mathcal{A}_\alpha+1)\alpha}{(1-x^2)^2}|u(X)|\,. \tag{3.3}$$

Because $\varphi(x)=|u(X)|$ we have that $\varphi\varphi'=\mathrm{Re}\{u'\bar{u}\}$ and

$$\varphi\varphi''\geq\mathrm{Re}\{u''\bar{u}\}=\mathrm{Re}\left\{\sum_{k=1}^{n}S^k_{11}F\frac{\partial u}{\partial z_k}\bar{u}\right\}+\mathrm{Re}\{S^0_{11}F\}\varphi^2\,.$$

However u satisfies the system (2.4) and using equation (3.3) one can conclude that

$$\varphi''\geq-\frac{\sqrt{n+1}(\mathcal{A}_\alpha+1)\alpha+C(n,\alpha)}{(1-x^2)^2}\,\varphi\,.$$

□

The following lemma establishes the main connection with convexity. By $\mathrm{Hess}(f)$ we denote the hessian operator of a smooth function f relative to the Bergman metric.

Lemma 3.3. *Let $F\in\mathcal{F}_\alpha$. There exists $\alpha_0>0$ sufficiently small such that λ_F is strictly convex in the Bergman metric of $\mathbb{B}^n$ if $\alpha\leq\alpha_0$. More specifically,*

$$\mathrm{Hess}(\lambda_F)\geq\beta^2\,\lambda_F\,g_B\,, \tag{3.4}$$

for $\beta=\beta(n,\alpha_0)>0$. In particular, λ_F can have at most one critical point in $\mathbb{B}^n$.

Proof. The inequality (3.4) entails showing that for each geodesic γ, parameterized by arc length in the Bergman metric, the function $(\lambda_F\circ\gamma)''\geq\beta^2\,\lambda_F$. In light of (3.1) and the fact that $\mathcal{F}_\alpha$ is linearly invariant, is suffices to show the required inequality at the origin for the Bergman geodesic $\gamma(t)=(x(t),0,\dots,0)$, that is, with $x(t)$ so that

$$x'(t)^2=\frac{(1-x(t)^2)^2}{n+1}\,.$$

Then

$$\frac{d\lambda_F}{dt}=\frac{d\lambda_F}{dx}\frac{dx}{dt}=\frac{d\lambda_F}{dx}\frac{1-x^2}{\sqrt{n+1}}$$
$$=\frac{1}{\sqrt{n+1}}\left(\frac{d|u_0|}{dx}(1-x^2)^{1/2}+|u_0|(1-x^2)^{-1/2}x\right)\,.$$

Because $u_0\neq0$ in $\mathbb{B}^n$, the previous lemma gives

$$
\begin{aligned}
\frac{d^2\lambda_F}{dt^2} &= \frac{1-x^2}{n+1}\left[\left(\frac{d^2|u_0|}{dx^2}\right)(1-x^2)^{1/2} + |u_0|(1-x^2)^{-3/2}\right] \\
&\geq \frac{(1-x^2)^{-1/2}}{n+1}|u_0|\left[1-\left(\sqrt{n+1}(\mathcal{A}_\alpha+1)\alpha + C(n,\alpha)\right)\right] .
\end{aligned}
$$

Because $C(n,\alpha) \to 0$ as $\alpha \to 0$, and since $\mathcal{A}_\alpha$ is decreasing in α, we conclude that

$$
\sqrt{n+1}(\mathcal{A}_\alpha+1)\alpha + C(n,\alpha) < 1 \tag{3.5}
$$

if α is sufficiently small. Then, with

$$
\beta^2(n,\alpha) = \frac{1}{n+1}\left[1-\left(\sqrt{n+1}(\mathcal{A}_\alpha+1)\alpha + C(n,\alpha)\right)\right] ,
$$

we obtain that at the origin

$$
\frac{d^2\lambda_F}{dt^2} \geq \beta^2(n,\alpha)\lambda_F ,
$$

as desired. □

For small values of α, the quantity $\beta(n,\alpha)\sqrt{n+1}$ is always less than 1, but can be made arbitrarily close to 1 by choosing α small enough. In particular, for given n, there exists α_0 sufficiently small so that for all $\alpha \in [0,\alpha_0]$

$$
\frac{1}{n} < \beta(n,\alpha)\sqrt{n+1} < 1 . \tag{3.6}
$$

To make this more precise, we observe that

$$
C(n,\alpha) \leq 6n^2\alpha^2 + 16\sqrt{n}\alpha ,
$$

so that, for example,

$$
C(n,\alpha) \leq \frac{1}{2}
$$

provided that $\alpha \leq \dfrac{1}{24n}$. In [7] it is shown that

$$
\mathcal{A}_\alpha \leq (n+1)\left[1+\sqrt{n+1}\,\alpha + C(n,\alpha)\right] ,
$$

so that in order to have, say,

$$
\sqrt{n+1}(\mathcal{A}_\alpha+1)\alpha + C(n,\alpha) \leq \frac{3}{4} ,
$$

it will be necessary to impose $\alpha \leq cn^{-\frac{3}{2}}$, for some constant c independent of n. Conditions (3.5) and (3.6) will be valid for α satisfying such a bound. In all future reference, we will consider

$$
\alpha_0 \leq cn^{-\frac{3}{2}} . \tag{3.7}
$$

Let $B_r = B(0,r) = \{z \in \mathbb{B}^n : |z| < s = s(r)\}$ be the ball centered at the origin of radius r in the Bergman metric, and let u be a solution of (3.1) for which $u \neq 0$ on B_r. We consider the modified function λ_{u,B_r} defined in B_r by

$$\lambda_{u,B_r}(z) = \frac{|u(z)|}{\sqrt{s^2 - |z|^2}}\,.$$

Together with this, it will be important to study the norm $||\mathcal{S}F||_{B_r}$ of the operator $\mathcal{S}F$ relative to the subdomain B_r.

Lemma 3.4. *If $z \in B(0,r)$ then*

$$\frac{s^2 - |z|^2}{1-|z|^2}\|v\|_r \leq \|v\| \leq \sqrt{\frac{s^2 - |z|^2}{1-|z|^2}}\|v\|_r\,,$$

where $\|\cdot\|_r$ is the Bergman norm at point $z \in B_r$.

Proof. We have that

$$\begin{aligned}
\|v\|_r^2 &= \frac{n+1}{(s^2-|z|^2)^2}\sum_{i,j=1}^n [(s^2 - |z^2|)\delta_{ij} + z_i\bar{z}_j]v_i\bar{v}_j \\
&= \frac{n+1}{(s^2-|z|^2)^2}\left(\sum_{i=1}^n (s^2 - |z|^2)|v_i|^2 + |z_iv_i|^2 + \sum_{i,j=1, i\neq j}^n z_i\bar{z}_jv_i\bar{v}_j\right) \\
&= \frac{n+1}{(s^2-|z|^2)^2}\left((s^2 - |z|^2)|v|^2 + |z_1v_1 + \cdots + z_nv_n|^2\right) \\
&\geq \frac{n+1}{s^2-|z|^2}|v|^2\,,
\end{aligned}$$

thus $(n+1)|v|^2 \leq (s^2 - |z|^2)\|v\|_r^2$. Now, the norm of v in the Bergman metric at $z \in B(0,r)$ with $|z| < r$ is

$$\begin{aligned}
\|v\|^2 &= \frac{n+1}{(1-|z|^2)^2}\sum_{i,j=1}^n [(1 - |z^2|)\delta_{ij} + z_i\bar{z}_j]v_i\bar{v}_j \\
&= \frac{n+1}{(1-|z|^2)^2}\left(\sum_{i=1}^n (1 - |z|^2)|v_i|^2 + |z_iv_i|^2 + \sum_{i,j=1, i\neq j}^n z_i\bar{z}_jv_i\bar{v}_j\right) \\
&= \frac{n+1}{(1-|z|^2)^2}\left(1 - |z|^2)|v|^2 + |z_1v_1 + \cdots + z_nv_n|^2\right) \\
&= \frac{n+1}{(1-|z|^2)^2}\left((1 - |z|^2)|v|^2 + \frac{(s^2-|z|^2)^2}{n+1}\|v\|_s^2 - |v|^2(s^2 - |z|^2)\right) \\
&= \frac{n+1}{(1-|z|^2)^2}\left(\frac{(s^2-|z|^2)^2}{n+1}\|v\|_r^2 + (1-s^2)|v|^2\right) \leq \frac{s^2-|z|^2}{1-|z|^2}\|v\|_r^2\,.
\end{aligned}$$

On the other hand, since $|z_1v_1+\cdots+z_nv_n|^2=\dfrac{(1-|z|^2)^2}{n+1}\|v\|^2-(1-|z|^2)|v|^2$ we have that

$$
\begin{aligned}
\|v\|_r^2 &= \frac{n+1}{(s^2-|z|^2)^2}\left((s^2-|z|^2)|v|^2+|z_1v_1+\cdots+z_nv_n|^2\right)\\
&= \frac{n+1}{(s^2-|z|^2)^2}\left((s^2-|z|^2)|v|^2+\frac{(1-|z|^2)^2}{n+1}\|v\|^2-|v|^2(1-|z|^2)\right)\\
&= \frac{n+1}{(s^2-|z|^2)^2}\left(\frac{(1-|z|^2)^2}{n+1}\|v\|^2+(s^2-1)|v|^2\right)\le\frac{(1-|z|^2)^2}{(s^2-|z|^2)^2}\|v\|^2\,.
\end{aligned}
$$

□

Lemma 3.5. *If* $||\mathcal{S}F||\le\alpha$ *then* $||\mathcal{S}F||_{B_r}\le\alpha\,$.

Proof. Using the above lemma a straightforward calculation gives that,

$$
\begin{aligned}
\|\mathcal{S}F(z)\|_{B_r} &= \sup_{\|v\|_r=1}\|\mathcal{S}F(z)(v,v)\|_r\\
&= \sup_{\|v\|_r=1}\left\|\mathcal{S}F(z)\left(\frac{v}{\|v\|},\frac{v}{\|v\|}\right)\right\|_r\|v\|^2\\
&\le \frac{1-|z|^2}{s^2-|z|^2}\|\mathcal{S}F(z)\|\|v\|^2\le\alpha.
\end{aligned}
$$

□

Lemma 3.6. *Let* $F\in\mathcal{F}_{\alpha_0}$ *and let* u *be a solution of (*3.1*) that does not vanish in* B_r*. Then* λ_{u,B_r} *is convex in the Bergman metric of* B_r*.*

Proof. By Lemma 2.1 there exists a locally injective mapping G, possibly with singularities, for which $u=(JG)^{-1/n+1}$ and $||\mathcal{S}G||\le\alpha_0$. By the assumption on u, G is regular in B_r and $||\mathcal{S}G||_{B_r}\le\alpha_0$ by the previous lemma. Hence the mapping $H(z)=G(rz)$ is locally biholomorphic in $\mathbb{B}^n$ and by invariance, $||\mathcal{S}H||\le\alpha_0$. It follows that λ_H is convex in $\mathbb{B}^n$, which implies the lemma because $\lambda_H(z)=r^{\frac{1}{n+1}}\lambda_{u,B_r}(rz)$. □

The following definition of the modified density function to sub-balls not centered at the origin will become clear from the proof of Lemma 3.7. Let $B=B(z_0,r)$ be the ball centered at $z_0\in\mathbb{B}^n$ of radius r in the Bergman metric, and let u be a solution of (3.1) with $u\neq0$ in B. Let σ be an automorphism of $\mathbb{B}^n$ taking B to B_r and z_0 to the origin. We define the modified density function associated with u in the ball B by

$$
\lambda_{u,B}=\frac{|u(z)||J\sigma|^{\frac{1}{n+1}}}{\sqrt{s^2-|\sigma(z)|^2}}\,,
$$

where, as before, $s=s(r)$ is the euclidean radius of B_r.

Lemma 3.7. *Let $F \in \mathcal{F}_{\alpha_0}$ and let u be a solution of (3.1) that does not vanish in $B = B(z_0, r)$. Then $\lambda_{u,B}$ is convex in the Bergman metric of B.*

Proof. If we write $u = (JG)^{-\frac{1}{n+1}}$ for some mapping G holomorphic in B with $||\mathcal{S}G|| \leq \alpha_0$, then $H = G \circ \sigma^{-1}$ becomes holomorphic in B_r and

$$u = (JG)^{-\frac{1}{n+1}} = (JH(\sigma))^{-\frac{1}{n+1}}(J\sigma)^{-\frac{1}{n+1}} = v(\sigma)(J\sigma)^{-\frac{1}{n+1}}\,,$$

where $v = (JH)^{-\frac{1}{n+1}}$. We see that

$$\lambda_{v,B_r}(\sigma(z)) = \frac{|v(\sigma(z))|}{\sqrt{s^2 - |\sigma(z)|^2}} = \frac{|u(z)||J\sigma|^{\frac{1}{n+1}}}{\sqrt{s^2 - |\sigma(z)|^2}} = \lambda_{u,B}(z)\,.$$

Because $||\mathcal{S}H|| = ||\mathcal{S}G|| \leq \alpha_0$, it follows from Lemma 3.6 that λ_{v,B_r} is convex in the Bergman metric of B_r, and therefore, that $\lambda_{u,B}$ is convex in the Bergman metric of B. □

We can now state two main results of this section.

Theorem 3.8. *Let $F \in \mathcal{F}_{\alpha_0}$ and let u be a solution of (3.1) with $u(z_0) = 1$ and $\nabla u(z_0) = 0$ for some $z_0 \in \mathbb{B}^n$. Then $u(z) \neq 0$ for all $z \in \mathbb{B}^n$.*

Proof. By the linear invariance of the family, we may assume that $z_0 = (0, \ldots, 0)$. There exists $r > 0$ such that $u \neq 0$ in B_r. Let r_1 be the supremum of such r and let $B = B_{r_1}$. We claim that $r_1 = \infty$. If not, there exists $z_1 \in \mathbb{B}^n$ with $|z_1| = s(r_1)$ for which $u(z_1) = 0$. By Lemma 3.6, the modified density function $\lambda_{u,B}$ is convex in B, and because $\nabla u(0) = 0$, the origin is an absolute minimum for $\lambda_{u,B}$ in B. Hence

$$|u(z)| \geq \frac{1}{r_1}\sqrt{s^2 - |z|^2}$$

for all $z \in B$. As $z \to z_1$ this last inequality leads to a contradiction because of the order of vanishing of $|u|$ in comparison with the right hand side. □

Theorem 3.9. *If $F \in \mathcal{F}_{\alpha_0}$ then F is univalent in $\mathbb{B}^n$.*

Proof. Suppose F is not injective. Then there exist linearly independent solutions $u_1, \ldots, u_n$ of (2.4) with $P_{ij}^k = S_{ij}^k F$ which vanish at $z_0 \neq z_1$ in $\mathbb{B}^n$. By linear invariance, we may take $z_0 = (0, \ldots, 0)$ and $z_1 = (a, 0, \ldots, 0)$, $0 < a < 1$. The gradients $\nabla u_1(z_0), \ldots, \nabla u_n(z_0)$ must be linearly independent, hence some linear combination $u = a_1 u_1 + \cdots + a_n u_n$ has $\nabla u(z_0) = (1, 0, \ldots, 0)$. Let $\Sigma = \{z \in \mathbb{B}^n : u(z) = 0\}$. Observe that because u is a non-trivial solution of (2.4), then $\nabla u(z)$ cannot vanish for $z \in \Sigma$, hence Σ is a smooth hypersurface. For $z_b = (b, 0, \ldots, 0)$, $0 < b < 1$, let $B(z_b, r(b))$ be the Bergman ball centered at z_b that is tangent to Σ at the origin z_0. It is known that the norm $||\mathcal{S}F||$ controls the second fundamental form of Σ in the Bergman metric, and at the origin we have the following. Let

$\gamma(t) = (z_1(t), \ldots, z_n(t))$ be an (euclidean) arclength parametrized line of curvature of Σ with $\gamma(0) = z_0$. Since $u(\gamma(t)) = 0$ we have

$$\mathrm{Hess}(u)(\gamma', \gamma') + \nabla u \cdot \gamma'' = 0\,,$$

and because γ is a line of curvature and $\nabla u(z_0) = (1, 0, \ldots, 0)$, we have that $\gamma''(0) = (k, 0, \ldots, 0)$. It follows that

$$|k| = \left| \sum_{i,j=1}^{n} \frac{\partial^2 u}{\partial z_i \partial z_j}(z_0) z_i'(0) z_j'(0) \right| .$$

In light of (2.4) we see that

$$|k| = \left| \gamma'(0)^t \mathbb{S}^1 F \gamma'(0) \right| .$$

Since $|\gamma'(0)| = 1$ in the euclidean norm, it follows finally that $|k| \leq (n+1)^{\frac{3}{2}} \alpha_0$. Because $\alpha_0 \leq cn^{-\frac{3}{2}}$, we can ensure, for small c, that Σ is flat enough at the origin so that the intersection of Σ and the closure of all balls $B(z_b, r(b))$ in a fixed (small) neighborhood of z_0 reduces to this point only. For b sufficiently small, then $u \neq 0$ in $B(z_b, b)$. Let b_1 be the supremum of all such b. We claim that $b_1 = 1/2$, contradicting the fact that $u(z_1) = 0$. If $b_1 < 1/2$ then $u \neq 0$ in $B_1 = B(z_{b_1}, b_1)$ but there exists $z_2 \in \overline{B_1}$ for which $u(z_2) = 0$. By the previous argument, z_2 is not the origin. We know that λ_{u,B_1} is convex in the Bergman metric of B_1, and again because of the orders of vanishing of $|u|$ at z_0, z_2 we conclude that $\lambda_{u,B_1}(z_0) = \lambda_{u,B_1}(z_2) = 0$. But then λ_{u,B_1} cannot be convex when restricted to the Bergman geodesic that terminates at z_0, z_2. This contradiction proves the theorem. □

4. Distortion and Continuity

In this section we will study continuous extension to the boundary of mappings in the class $\mathcal{F}_{\alpha_0}$ which are quasiregular in the ball. As a complement to Theorem 3.9, we will show that the extension remains injective in the closed ball, setting up the stage in the final section for the construction of the homeomorphic extension. A mapping $F \in \mathcal{F}_{\alpha_0}$ will be said to belong to $\mathcal{F}^*_{\alpha_0}$ if $\nabla(JF)(0) = 0$.

Theorem 4.1. *Let $F \in \mathcal{F}^*_{\alpha_0}$. Then*

$$|JF(z)| \leq \frac{2}{(1-|z|)^{\gamma}}\,, \tag{4.1}$$

where

$$\gamma = \gamma(n, \alpha_0) = \frac{n+1}{2}\left(1 - \beta(n, \alpha_0)\sqrt{n+1}\right) .$$

If, in addition, F is quasiregular in $\mathbb{B}^n$ then there exists $C > 0$ such that

$$||DF(z)|| \leq \frac{C}{(1-|z|)^{\frac{\gamma}{n}}} . \tag{4.2}$$

In particular, $F(\mathbb{B}^n)$ is a bounded domain.

Proof. Because λ_F is strictly convex in the Bergman metric with the conditions at the origin $\lambda_F(0)=1$, $\nabla\lambda_F(0)=0$ it follows from (3.4) that

$$\lambda_F(z)\geq\cosh\left(\beta d(0,z)\right)\,,$$

where $d(0,z)$ is the Bergman distance

$$d(0,z)=\frac{\sqrt{n+1}}{2}\log\frac{1+|z|}{1-|z|}\geq\frac{\sqrt{n+1}}{2}\log\frac{1}{1-|z|}\,.$$

Then (4.1) obtains from the definition of λ_F, together with $\cosh(t)\geq\frac{1}{2}e^t$ and

$$\lambda_F(z)\leq\frac{1}{(1-|z|)|JF|^{\frac{1}{n+1}}}\,.$$

The estimate (4.2) follows from (4.1) directly if F is quasiregular in $\mathbb{B}^n$. Because $\frac{\gamma}{n}\in(0,1)$, we deduce the final claim that the image is bounded since the right hand side in (4.2) is integrable on $[0,1)$. $\square$

Remarks: Because the right hand side in (4.1) is integrable in the ball, the image $F(\mathbb{B}^n)$ has finite volume regardless of quasiregularity.

It is interesting to note that the first part of Theorem 4.1 provides better estimates on the jacobian than those coming from the order of the linearly invariant family [28].

Corollary 4.2. *Let $F\in\mathcal{F}^*_{\alpha_0}$ be quasiregular in $\mathbb{B}^n$. Then F admits a Hölder continuous extension to $\overline{\mathbb{B}^n}$.*

Proof. The proof follows directly from the estimate (4.2) and Lemma 8.5.4 in [10]. The resulting exponent of Hölder continuity is $1-\frac{\gamma}{n}$. $\square$

It is natural to inquire about the injectivity of the mapping F in the closed ball. To answer this question it will be necessary to establish first an important geometric property of the image domain. Suppose $F\in\mathcal{F}_{\alpha_0}$ is not normalized to have $\nabla(JF)(0)=0$. If M is a Möbius transformation of the form

$$M(w)=\left(\frac{w_1}{1-a\cdot w},\ldots,\frac{w_n}{1-a\cdot w}\right)\,,\tag{4.3}$$

where $a\cdot w=a_1w_1+\cdots+a_nw_n$, then the mapping $H=M\circ F$ will satisfy $H(0)=0$ and $DH(0)=\mathrm{Id}$. By choosing $a=(a_1,\ldots,a_n)=-\frac{1}{n+1}\nabla(JF)(0)$ we can achieve that $\nabla(JH)(0)=0$. Because of the invariance under Möbius transformations, $\mathcal{S}H=\mathcal{S}F$, although H will cease to be regular if there are points $z\in\mathbb{B}^n$ for which $a\cdot F(z)=1$. We claim, nevertheless, that this cannot occur. The calculation given in the proof of Lemma 2.1 shows that, at points where $a\cdot F(z)=1$, one would have $u = (JH)^{-\frac{1}{n+1}} = 0$. But $u = (JH)^{-\frac{1}{n+1}}$ is a solution of (3.1) with $u(0) = 1, \nabla u(0) = 0$, and therefore, by Theorem 3.8, cannot vanish in $\mathbb{B}^n$. This proves our claim. We have concluded that for any mapping $F \in \mathcal{F}_{\alpha_0}$ there exists a Möbius transformation M such that $H = M \circ F \in \mathcal{F}^*_{\alpha_0}$. In particular, the image $F(\mathbb{B}^n)$ does not meet the hyperplane $a \cdot w = 1$, denoted by $\mathcal{H}(0)$. In the case that H is also quasiregular in $\mathbb{B}^n$, it will be bounded by Theorem 4.1, which implies that the closure of $F(\mathbb{B}^n)$ does not meet $\mathcal{H}(0)$.

We define the hyperplane $\mathcal{H}(z_0)$ for a generic $z_0 \in \mathbb{B}^n$ by considering Koebe transforms. For $z_0 \in \mathbb{B}^n$ and $F \in \mathcal{F}_\alpha$, let σ be an automorphism of $\mathbb{B}^n$ with $\sigma(0) = z_0$. Then

$$G(z) = D\sigma(0)^{-1} DF(z_0)^{-1} \left[F(\sigma(z)) - F(z_0)\right] \tag{4.4}$$

is called a Koebe transform of F. It is normalized to have $G(0) = 0, DG(0) = \mathrm{Id}$, and has $\|\mathcal{S}G\| = \|\mathcal{S}F\|$. As we have seen, the composition $H = M \circ G$ with

$$M(w) = \left(\frac{w_1}{1 - a \cdot w}, \ldots, \frac{w_n}{1 - a \cdot w}\right)$$

and $a = a(z_0, F) = -\frac{1}{n+1}\nabla(JG)(0)$ produces a critical point for λ_H at $z = 0$. The hyperplane $\mathcal{H}(z_0)$, omitted by $F(\mathbb{B}^n)$, will emerge as a consequence of the fact that the image $G(\mathbb{B}^n)$ omits the hyperplane $a(z_0, F) \cdot z = 1$. To make this precise, observe that

$$JG(z) = J\sigma(0)^{-1} JF(z_0)^{-1} JF(\sigma z) J\sigma(z)\,,$$

hence

$$\frac{\nabla(JG)}{JG}(z) = \frac{\nabla(JF)}{JF}(\sigma(z)) D\sigma(z) + \frac{\nabla(J\sigma)}{J\sigma}(z)\,,$$

which at the origin gives

$$\nabla(JG)(0) = \frac{\nabla(JF)}{JF}(z_0) D\sigma(0) + \frac{\nabla(J\sigma)}{J\sigma}(0)\,.$$

If $\mathcal{K}(z_0)$ denotes the hyperplane $a(z_0, F) \cdot w = 1$ omitted by $G(\mathbb{B}^n)$, then we see that $F(\mathbb{B}^n)$ omits the hyperplane given by

$$\mathcal{H}(z_0) = F(z_0) + DF(z_0) D\sigma(0) \mathcal{K}(z_0)\,. \tag{4.5}$$

We remark that, although the automorphism σ is unique only up to precomposition with a rotation of the ball, the resulting hyperplane $\mathcal{H}(z_0)$ only depends on the point z_0 and not on the particular choice of automorphism σ taking the origin to z_0. We will omit the calculation. Another important property that follows from derivation of $\mathcal{H}(z_0)$ is that an inversion of F with respect to this hyperplane produces a critical point of the resulting convex function at z_0.

Because
$$\frac{\overline{a(z_0,F)}}{||a(z_0,F)||^2} \in \mathcal{K}(z_0)\,,$$
we see that
$$\mathcal{E}(z_0) = F(z_0) + DF(z_0)D\sigma(0)\frac{\overline{a(z_0,F)}}{||a(z_0,F)||^2} \in \mathcal{H}(z_0)\,,$$
and hence $\mathcal{E}(z_0)$ is a value omitted by $F(\mathbb{B}^n)$ for every $z_0 \in \mathbb{B}^n$. It will be relevant for the extension in the next section that the hyperplane $\mathcal{H}(z_0)$ remains disjoint even from the closure $\overline{F(\mathbb{B}^n)}$ when F is quasiregular.

Remark: When $n = 1$, then the hyperplanes above reduce to single points and $\mathcal{E}(z)$ becomes
$$f(z) + \frac{(1-|z|^2)f'(z)}{\bar{z} - \frac{1}{2}(1-|z|^2)\frac{f''}{f'}(z)}\,,$$
which is the Ahlfors-Weill formula that extends mappings f in the Nehari class.

Lemma 4.3. *Let $F \in \mathcal{F}^*_{\alpha_0}$ be quasiregular. Then for each $z_0 \in \mathbb{B}^n$ the hyperplane $\mathcal{H}(z_0)$ is disjoint from the closure $\overline{F(\mathbb{B}^n)}$.*

Proof. Suppose not. By the analysis leading to the definition of $\mathcal{H}(z_0)$, this means that the hyperplane $a \cdot w = 1$ is not disjoint from $\overline{G(\mathbb{B}^n)}$, where G is the Koebe transform of F in (4.4) and $a = a(z_0, F)$. Hence, there exists $\zeta_0 \in \partial\mathbb{B}^n$ such that $a \cdot G(\zeta_0) = 1$. Because $H \in \mathcal{F}^*_{\alpha_0}$, then (4.1) implies that
$$|JH| \le \frac{1}{(1-|z|)^\gamma}\,,$$
but at the same time,
$$JH(z) = \frac{JG(z)}{(1 - a\cdot G(z))^{n+1}}\,.$$
The mapping G inherits quasiregularity from F, and so,
$$\frac{||DG(z)||}{|1-a\cdot G(z)|^{1+\frac{1}{n}}} \le \frac{1}{(1-|z|)^{\frac{\gamma}{n}}}\,.$$
The function $\phi(z) = (1 - a \cdot G(z))^{-\frac{1}{n}}$ is holomorphic in $\mathbb{B}^n$ and has $|\phi(\zeta_0)| = \infty$, which leads to contradiction since
$$|\nabla\phi(z)| \le \frac{||DG(z)||}{|1-a\cdot G(z)|^{1+\frac{1}{n}}} \le \frac{1}{(1-|z|)^{\frac{\gamma}{n}}}\,,$$
with the right hand side integrable. This finishes the proof. $\square$

Lemma 4.4. *Let $F \in \mathcal{F}^*_{\alpha_0}$ be quasiregular, and let $\{z_n\} \subset \mathbb{B}^n$ be a sequence with $z_n \to \zeta \in \partial\mathbb{B}^n$. Then $\mathcal{E}(z_n) \to F(\zeta)$.*

Proof. We will show that $\mathcal{E}(z) - F(z) \to 0$ as $|z| \to 1$, which will prove the lemma because F is continuous up to the boundary. To this end we will show that

$$||\mathcal{E}(z) - F(z)|| = O\left((1-r)^{\frac{1}{2}\left(\beta\sqrt{n+1}-\frac{1}{n}\right)}\right) , \tag{4.6}$$

where $r = |z|$ and $\beta = \beta(n, \alpha)$ as in Lemma 3.3. Fix $z \in \mathbb{B}^n$. By rotating F we may assume that $z = (z_1, 0, \ldots, 0)$. Then

$$||\mathcal{E}(z) - F(z)|| \leq ||DF(z)|| \frac{||D\sigma(0)(a(z))||}{||a(z)||^2} , \tag{4.7}$$

where we use $a(z)$ for $a(z, F)$. The term $||DF(z)||$ is $O(|JF(z)|^{\frac{1}{n}})$ because of quasiregularity. Next,

$$a(z) = -\frac{1}{n+1}\nabla(JG)(0) = -\frac{1}{n+1}\left[\frac{\nabla(JF)}{JF}(z)D\sigma(0) + \frac{\nabla(J\sigma)}{J\sigma}(0)\right] .$$

The mapping σ may be taken of the form

$$\sigma(w) = \left(\frac{w_1 + z_1}{1 + \bar{z}_1 w_1}, \frac{\sqrt{1-|z_1|^2}w_2}{1 + \bar{z}_1 w_1}, \ldots, \frac{\sqrt{1-|z_1|^2}w_n}{1 + \bar{z}_1 w_1}\right) ,$$

hence

$$D\sigma(0) = \begin{pmatrix} 1-|z_1|^2 & 0 & \cdots & 0 \\ 0 & \sqrt{1-|z_1|^2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sqrt{1-|z_1|^2} \end{pmatrix} ,$$

and

$$J\sigma(w) = \frac{(1-|z_1|^2)^{\frac{1}{2}(n+1)}}{(1+\bar{z}_1 w_1)^{n+1}} ,$$

from which we obtain

$$\frac{\nabla(J\sigma)}{J\sigma}(0) = -(n+1)\left(\bar{z}_1, 0, \ldots, 0\right) .$$

Then

$$\begin{aligned} a(z) &= (\bar{z}_1, 0, \ldots, 0) - \frac{1}{n+1}\frac{\nabla(JF)}{JF}(z)D\sigma(0) \\ &= \left[\left(\frac{\bar{z}_1}{1-|\bar{z}_1|^2}, 0, \ldots, 0\right) - \frac{1}{n+1}\frac{\nabla(JF)}{JF}(z)\right] D\sigma(0) \end{aligned}$$

$$= [A + B]\, D\sigma(0) = (1 - |\bar{z}_1|^2)A + \sqrt{1 - |\bar{z}_1|^2}B\,,$$

where $A = (a_1, 0, \ldots, 0)$ and $B = (0, b_2, \ldots, b_n)$. Then

$$D\sigma(0)(a(z)) = (1 - |\bar{z}_1|^2)^2 A + (1 - |\bar{z}_1|^2)B\,.$$

This leads us to estimate the quantity

$$\frac{||(1 - |\bar{z}_1|^2)^2 A + (1 - |\bar{z}_1|^2)B||}{||(1 - |\bar{z}_1|^2)A + \sqrt{1 - |\bar{z}_1|^2}B||^2}\,,$$

or equivalently,

$$\frac{(1 - |z_1|^2)^2||A|| + (1 - |z_1|^2)||B||}{(1 - |z_1|^2)^2||A||^2 + (1 - |z_1|^2)||B||^2} = \frac{ax + by}{ax^2 + by^2} = \phi(x, y)\,,$$

where $a = b^2 = (1 - |\bar{z}_1|^2)^2$ and $x = ||A||, y = ||B||$. For a, b, x fixed, the maximal value of $\phi(x, y)$ when $y \geq 0$ occurs when

$$y = \frac{\sqrt{a}\,x}{\sqrt{a} + \sqrt{a + b}} \sim \sqrt{1 - |z_1|^2} \quad , \quad |z_1| \to 1\,,$$

with corresponding maximal value for ϕ asymptotically equal to

$$\rho = \frac{1}{2\sqrt{1 - |z_1|^2}\, x}$$

for $|z_1|$ close to 1. The final step requires to find a lower bound for $x = ||A||$, which will arise as a consequence of (3.4). Indeed, (3.4) together with the critical point at the origin, imply an exponential growth for λ_F along the ray $[\zeta, 0, \ldots, 0)$ in the Bergman metric. Therefore,

$$(1 - |\zeta|^2)\, |\partial_\zeta \lambda_F(\zeta, 0, \ldots, 0)| \geq \beta \lambda_F(\zeta, 0, \ldots, 0)\,.$$

By evaluating at $\zeta = z_1$ we see that

$$x = ||A|| \geq \frac{\beta}{1 - |z_1|^2}\,,$$

which gives

$$\rho \leq \frac{\sqrt{1 - |z_1|^2}}{2\beta}\,.$$

Finally, in (4.7) we see that

$$||\mathcal{E}(z) - F(z)|| \leq C|JF(z)|^{\frac{1}{n}}\sqrt{1 - |z_1|^2} = C\,\frac{\lambda_F(z)^{-\frac{n+1}{n}}}{(1 - |z_1|^2)^{\frac{1}{2n}}}$$

$$\leq C\,\frac{\lambda_F(z)^{-1}}{(1-|z_1|^2)^{\frac{1}{2n}}} \leq C\,\frac{e^{-\beta d_B(0,z)}}{(1-|z_1|^2)^{\frac{1}{2n}}}\,,$$

where $d_B(0,z) = \frac{\sqrt{n+1}}{2}\log\frac{1+|z|}{1-|z|}$ is the Bergman distance to the origin. This finally gives

$$||\mathcal{E}(z) - F(z)|| \leq C\,(1-|z_1|^2)^{\frac{1}{2}(\beta\sqrt{n+1}-\frac{1}{n})}\,,$$

which proves (4.6). □

Theorem 4.5. *If $F \in \mathcal{F}^*_{\alpha_0}$ is quasiregular then $F(\mathbb{B}^n)$ is weakly linearly convex.*

Proof. Let $F(\zeta)$ be a boundary point of $F(\mathbb{B}^n)$, $\zeta \in \partial\mathbb{B}^n$, and consider the sequence of points $z_n = r_n\zeta \in \mathbb{B}^n$ with $r_n \to 1$. The family of hyperplanes $\mathcal{H}(z_n)$ contains the sequence of points $\mathcal{E}(z_n)$ which converge to $F(\zeta)$. It follows that some subsequence of hyperplanes converge to a hyperplane $\mathcal{H}(\zeta)$ that contains the point $F(\zeta)$. The hyperplane $\mathcal{H}(\zeta)$ must be omitted by $F(\mathbb{B}^n)$ since $F(\mathbb{B}^n) \cap \mathcal{H}(z_n) = \emptyset$ for every n. □

Even though quasiregularity becomes an essential hypothesis for the univalence on the boundary and for an actual gluing of $F(\mathbb{B}^n)$ with the extension, we can draw the following corollary.

Corollary 4.6. *If $F \in \mathcal{F}^*_{\alpha_0}$ then $F(\mathbb{B}^n)$ is weakly linearly convex.*

Proof. For $0 < r < 1$ we consider the mappings $F_r(z) = \frac{1}{r}F(rz)$. The mappings F_r have $\nabla(JF_r)(0) = 0$, and Lemma 3.5 then shows that $F_r \in \mathcal{F}^*_{\alpha_0}$. Since F_r are quasiregular, we deduce from Theorem 4.5 that the images $F(r\mathbb{B}^n)$ are weakly linearly convex, from which the corollary follows. □

We can now state the last result of this section.

Theorem 4.7. *Let $F \in \mathcal{F}^*_{\alpha_0}$ be quasiregular in $\mathbb{B}^n$. Then F is univalent in $\overline{\mathbb{B}^n}$.*

Proof. We know that F extends continuously to the closed ball. Suppose F fails to be injective in $\overline{\mathbb{B}^n}$. Then there are distinct points $\zeta_1, \zeta_2 \in \partial\mathbb{B}^n$ for which $F(\zeta_1) = F(\zeta_2) = w_0 \in \partial\Omega$, $\Omega = F(\mathbb{B}^n)$. Let $H = \{a_0 + a_1w_1 + \cdots + a_nw_n = 0\}$ be a hyperplane of support to Ω at w_0, and let T be the Möbius transformation

$$T(w) = \left(\frac{w_1}{a_0 + a_1w_1 + \cdots + a_nw_n}, \ldots, \frac{w_n}{a_0 + a_1w_1 + \cdots + a_nw_n}\right).$$

The mapping $G = T \circ F$ is holomorphic in $\mathbb{B}^n$ because of the choice of the hyperplane H, but $G(\zeta_1), G(\zeta_2)$ are the point at infinity. Consider the convex function λ_G along the (Bergman) geodesic Γ joining ζ_1 and ζ_2. Because of (3.4), then λ_G

will exhibit exponential growth along Γ at least in the direction of one of the endpoints, say, in the direction of ζ_1. Then, as in the proof of Theorem 4.1, we will have

$$|JG(z)| \leq \frac{C}{(1-|z|)^{\gamma}}$$

along Γ in the direction of ζ_1, for some constant C. But

$$|JG(z)| = \frac{|JF(z)|}{|a_0 + a\cdot F(z)|^{n+1}} \quad , \; a = (a_1, \ldots, a_n)\,,$$

and using the quasiregularity of F we obtain

$$\frac{||DF(z)||}{|a_0 + a\cdot F(z)|^{1+\frac{1}{n}}} \leq \frac{C_1}{(1-|z|)^{\frac{\gamma}{n}}}\,.$$

But the function

$$\phi(z) = \frac{1}{(a_0 + a\cdot F(z))^{\frac{1}{n}}}$$

is holomorphic in $\mathbb{B}^n$ and tends to infinity at ζ_1, which leads to a contradiction because

$$|\nabla\phi(z)| \leq \frac{||a||||DF(z)||}{|a_0 + a\cdot F(z)|^{1+\frac{1}{n}}} \leq \frac{C_2}{(1-|z|)^{\frac{\gamma}{n}}}\,,$$

and the right hand is integrable along Γ. This finishes the proof. $\Box$

Remark: The same method of proof shows that, under the same assumptions as in Theorem 4.6, a hyperplane of support at a boundary point of $F(\mathbb{B}^n)$ can meet the boundary only at that point.

5. Extensions

In this section, we collect our previous results to obtain the extension E_F to $\mathbb{C}^n$ of mappings $F \in \mathcal{F}_{\alpha_0}$ that are quasiregular in $\mathbb{B}^n$. For points $z \notin \overline{\mathbb{B}^n}$, the extension will take values in the set of complex hyperplanes, but will remain a homeomorphism in the sense that that it assigns in a continuous (even real analytic) fashion a unique hyperplane for each such z. For $z \in \mathbb{C}^n$ we recall the notation $z^* = z/|z|^2$, and recall the hyperplane defined in (4.5).

Theorem 5.1. *Let $F \in \mathcal{F}^*_{\alpha_0}$ be quasiregular and consider*

$$E_F(z) = \begin{cases} F(z) & , \; |z| \leq 1 \\ \mathcal{H}(z^*) & , \; |z| > 1 \end{cases} \qquad (5.1)$$

Then E_F is a homeomorphic extension of F that glues $F(\overline{\mathbb{B}^n})$ to a complementary domain in the projective dual of $\mathbb{C}^n$.

Proof. By Theorem 4.6, we know that F is univalent in $\overline{\mathbb{B}^n}$. On the other hand, by Lemma 4.3, the collection of hyperplanes $\mathcal{H}(z^*)$, $|z| > 1$, is disjoint from $F(\overline{\mathbb{B}^n})$. Furthermore, since an inversion of F with respect to $\mathcal{H}(z^*)$ produces a critical point of the density function λ_F at z^*, we see from the strict convexity that such hyperplanes must be distinct for different z^*. This shows that E_F is injective. By the nature of the explicit formula in (4.5), it is clear that the extension in (5.1) is also continuous (and real analytic).

We comment finally on the location of the hyperplanes $\mathcal{H}(z^*)$ when $|z| > 1$ is very large. To that extent, we use equation (4.5) with $z_0 = z^*$ near the origin. Simple calculations show that $a(z_0, F) = O(z_0)$, hence the hyperplane $\mathcal{K}(z_0)$ given by $a \cdot w = 1$ is far away from the origin. It follows now from (4.5) that $\mathcal{H}(z_0)$ itself will also be far away from the origin. □

References

[1] Ahlfors, L.V., Sufficient conditions for quasi-conformal extension. Discontinuous groups and Riemann surfaces, *Annals of Math. Studies* **79** (1974), 23-29.

[2] Ahlfors, L.V. and Weill, G., A uniqueness theorem for Beltrami equations, *Proc. Amer. Math. Soc.* **13** (1962) , 975-978.

[3] Andersson, A., Passare, M. and Sigurdsson, R., Complex Convexity and Analytic Functionals, *Progress in Mathematics* **225**, Birhäuser, Basel, 2004.

[4] Becker, J., Löwnersche Differentialgleichung und quasi-konform fort setzbare schlichte Funktionen, *J. Reine. Angew. Math.* **225** (1972), 23-43.

[5] Chuaqui, M. and Osgood, B., Sharp distortion theorems associated with the Schwarzian derivative, *Jour. London Math.Soc.* **48** (1993), 289-298.

[6] Chuaqui, M. and Osgood, B., Ahlfors-Weill extensions of conformal mappings and critical points of the Poincaré metric, *Comment. Math. Helv.* **69** (1994), 659-668.

[7] Chuaqui, M. and Hernández, R., The order of a linearly invariant family in $\mathbb{C}^n$, to appear in J. Math. Anal. Appl. (arxiv.org/abs/1206.0642).

[8] Duren, P., Univalent Functions, *Springer-Verlag*, 1983.

[9] Gehring, F.W. and Pommerenke, Ch., On the Nehari univalence criterion and quasicircles, *Comment. Math. Helv.* **59** (1984), 226-242.

[10] Graham, I. and Kohr, G., Geometric function theory in one and higher dimensions, *Pure and Applied Math.* **255**, Marcel Dekker, 2003.

[11] Hamada, H. and Kohr, G., The growth theorem and quasiconformal extension of strongly spirallike mappings of type α, *Complex Variables* **44** (2001), 281-297.

[12] Hamada, H. and Kohr, G., Loewner chains and quasiconformal extension of holomorphic mappings, *Ann. Polon. Math.* **81** (2003), 85-100.

[13] Hernández, R., Schwarzian derivatives and a linearly invariant family in $\mathbb{C}^n$, *Pacific J. Math.* **228** (2006), 201-218.

[14] Hernández, R., Schwarzian derivatives and some criteria for univalence in $\mathbb{C}^n$, *Complex Variables* **52** (2007), 397-410.

[15] Hörmander, L., Notions of Convexity, *Modern Birkäuser Classics*, Birkäuser, Boston, 2007.

[16] Kikuchi, K., Starlike and convex mappings in several complex variables, *Pacific J. Math.* **44** (1973), 569-580.

[17] Krantz, S., Function Theory of Several Complex Variables, *Pure and Applied Math.*, Wiley, 1982.

[18] Molzon, R. and Mortensen, K.P., Differential operators associated with holomorphic mappings, *Annals of Global Analysis and Geometry* **12** (1994), 291-304.

[19] Molzon, R. and Mortensen, K.P., The Schwarzian derivative for maps between manifolds with complex projective connections, *Trans. Amer. Math. Soc.* **348** (1996), 3015-3036.

[20] Molzon, R. and Mortensen, K.P., Univalence of Holomorphic Mappings, *Pacific J. Math.* **180** (1997), 125-133.

[21] Molzon R. and Tamanoi H., Generalized Schwarzians in several variables and Möbius invariant differential operators, *Forum Math.* **14** (2002), 165-188.

[22] Nehari, Z., The Schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc.* **55** (1949), 545-551.

[23] Nehari, Z., Some criteria of univalence, *Proc. Amer. Math. Soc.* **5** (1954), 700-704.

[24] Nehari, Z., A property of convex conformal maps, *J. Anal. Math.* **30** (1976), 390-393.

[25] Oda, T., On Schwarzian Derivatives in several variables (in Japanese). *Kokyuroku of R.I.M., Kyoto Univ.* **226** (1974).

[26] Pfaltzgraff, J.A., Subordination chains and univalence of holomorphic mappings in $\mathbb{C}^n$, *Math. Ann.* **210** (1974), 55-68.

[27] Pfaltzgraff, J.A., Subordination chains and quasiconformal extension of holomorphic maps in $\mathbb{C}^n$, *Ann. Acad. Sci. Fenn. Ser. AI Math.* **1** (1975), 13-25.

[28] Pfaltzgraff, J.A., Distortion of locally biholomorphic maps of the n-ball, *Complex Variables* **33** (1997), 239-253.

[29] Pommerenke, Ch., Linear-invariante Familien analytischer Funktionen I, *Math. Ann.* **155** (1964), 108-154.

[30] Suffridge, T., Starlikeness, convexity and other geometric properties of holomorphic maps in higher dimensions. Lecture Notes in Math. *Springer-Verlag, New York* **599** (1976), 146-159.

[31] Yoshida, M., Canonical forms of some system of linear partial differential equations, *Proc. Japan Acad.* **52** (1976), 473-476.

[32] Yoshida, M., Orbifold-Uniformizing Differential Equations, *Math. Ann.* **267** (1984), 125-142.

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